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# Perturbation of low-frequency underwater acoustics by gravity waves

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A body immersed in an ocean of large depth is assumed to vibrate and to radiate a time-harmonic acoustic field of small amplitude in the presence of gravity waves of small amplitude. Assuming both waves to have lengths of the same order (which in practice corresponds to very low acoustic frequencies) it is shown that the diffraction of acoustic waves by the corrugated free surface generates a second-order acoustic pressure field $p_2$. The computation of $p_2$ involves a difficulty: a non-homogeneous Dirichlet condition to be satisfied on the mean free surface up to infinity which implies the absence of any clear indication about the condition that should be imposed at infinity to have a well-posed problem. In order to get an insight into this difficult problem the simple case of a point source is studied. We first judiciously choose one solution and then show it is the physical solution using a limiting-amplitude procedure. Coming back to the general case of a vibrating body the calculation of $p_2$ is split into two successive steps: the first one consists in computing an explicit convolution product via numerical methods of integration, the second one is a standard radiation problem that is solved using a method coupling a Green integral representation and finite elements. A peak of the second-order pressure appears just above the vibrating body.

The same concepts also apply to other second-order scattering problems, such as the sea-keeping of weakly immersed submarines.

## 1. Introduction

Let us first consider a body immersed in a quiet ocean of large depth, vibrating in a known time-harmonic way and thus radiating an acoustic pressure field $p_1$ of small amplitude referred to as 'the first-order' acoustic pressure: this pressure field will satisfy a Helmholtz equation in the liquid domain, together with given Neumann data on the mean position $\Gamma$ of the hull of the body, a homogeneous Dirichlet condition on the free surface $z = 0$ and the standard radiation condition; good methods are available to solve numerically such a problem.

Let us then suppose that there are gravity waves. The question is: how will the acoustic propagation be perturbed by their presence (see figure 1)? In a first approximation, the influence of gravity waves is taken into account only by the shape of the free surface which can be considered as fixed at the acoustic time-scale. When

† Daniel Euvrard has laid with his legendary enthusiasm the foundations of the present work. He started writing this article, but his passion for flying took him in an accident on July 10, 1994.

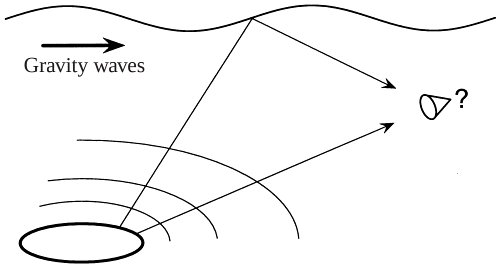


FIGURE 1. The problem.

this free surface is periodic (the case of a plane swell), the problem comes within the theory of diffraction gratings. A possible direct approach could be to use classical integral equation techniques involving the Green function of the grating. But such a function is extremely difficult to compute. That is why we adopt an asymptotic point of view by assuming that the amplitude $\mathscr{A}$ of the swell is small. In § 2 we shall construct a physical pattern where the influence of gravity waves appears as a 'second order' acoustic pressure. The difficulty in the solution of this problem is the absence of an explicit condition at infinity, which leads to an ill-posed problem. If we substitute a point source for the body the new problem is still ill-posed but we can calculate explicitly all its solutions (see § 3). Then we will judiciously choose one and justify it by applying a *limiting-amplitude* process. In § 4, coming back to the problem with the body, we will choose a solution, using the same argument as in the case of a point source (exhibited in § 3). We will then propose a way to solve numerically the problem and we will present some numerical results. Finally in § 5, the theoretical justification of this method will be sketched.

This method is a continuation of the work initiated by Mechiche Alami (1992) and Euvrard & Mechiche Alami (1992). It is not only useful in underwater acoustics (its application in this context seems to be new), but also in the applications it may have to ship naval hydrodynamics. Indeed the second-order sea-keeping of off-shore structures and weakly immersed submarines is a problem of major importance. At least for submarines it can be presented in a coherent asymptotic theory. For instance if the submarine oscillates with a frequency $\omega/(2\pi)$ in calm water second-order terms of frequencies 0 and $\omega/\pi$ appear. The velocity potential corresponding to the latter must satisfy the Laplace equation, together with a Neumann condition on the hull and the following non-homogeneous Robin–Fourier condition on the mean free surface:

$$\frac{\partial \varphi_2}{\partial z} + \frac{4\,\omega^2}{g}\,\varphi_2 = q \quad \text{for} \quad z = 0,$$

where $q$ is a known function which depends on the first-order potential $\varphi_1$. The difficulty concerning this non-homogeneous boundary condition up to infinity, and consequently the absence of a clear condition at infinity, is the same as in our problem of acoustics. So the same concepts can be applied: our approach follows the same ideas as in the paper by Sclavounos (1988) who deals with three-dimensional sea-keeping problem. His method consists in introducing some 'second-order Green functions' which are derived (by means of a horizontal Fourier transform) using the same limiting-amplitude technique as in § 3. For a floating or immersed body, these functions allow the second-order velocity potential to be expressed as a boundary integral. As regards the numerical implementation, this approach has been developed in a slightly different form (which avoids introducing these functions) by Papin (1990) and Friis, Grue & Palm (1991) in the two-dimensional situation, and more recently by Bellier (1997) for the three-dimensional sea-keeping of submarines. The leading idea is to search for a *particular* solution of the second-order problem by splitting it into a 'free' velocity potential which ignores the body but satisfies the proper free-surface condition, and a 'correction' term satisfying a homogeneous free-surface condition (like the first-order potential): the former is obtained by Fourier transform whereas the latter can be computed by classical integral techniques. Unfortunately, usual fast Fourier transform algorithms are not adapted for the numerical approximation of the 'free' potential. The precise numerical procedures described in the present paper are partly based on Fourier series and Hankel transforms. They were applied by Bellier (1997) in hydrodynamics.

## 2. Problem formulation

In this section, we are interested in constructing a system of equations which models the influence of gravity waves on the acoustic waves radiated from an immersed body.

We study the case where wavelengths of both gravity and acoustic waves are of the same order. For usual swells, this implies on one hand that low-frequency acoustics is considered, and on the other hand that the acoustic period $T_a$ is much smaller than the gravity wave period $T_g$. Indeed, suppose for example that the acoustic and gravity wavelengths are equal, and denoted by $\Lambda$. Then $\Lambda = c_a T_a = c_g T_g$, where $c_a$ is the sound speed in water and $c_g = gT_g/2\pi$ the celerity of gravity waves ($g$ is the gravitational acceleration). As $c_a = 1500\,\mathrm{m\,s^{-1}}$, we infer that $T_g^2 \approx 10^3 \times T_a$ (where both periods are expressed in s): for instance, if $T_a = 0.1\,\mathrm{s}$ (which corresponds to $\Lambda = 150\,\mathrm{m}$), we have $T_g \approx 10\,\mathrm{s}$. Hence, in a first approximation, we can consider that gravity waves are fixed as far as the acoustic time-scale is considered. Thus we study an acoustic problem in a domain bounded by a corrugated free surface.

This way of presenting the problem may be seen as a simplification of a more general approach which consists in expanding the physical variables according to two parameters representing the amplitudes of the perturbations of acoustic and hydrodynamic phenomena in the conservation equations (see Champy-Doutreleau 1998).

We suppose the ocean to be an homogeneous ideal fluid. Let $\Omega$ be a three-dimensional fluid domain, infinitely deep and delimited by the hull $\Gamma$ of the body and the corrugated free surface $FS$. We assume that the equation of this free surface is known:

$$z = \mathscr{A}\,\eta(x, y)$$

where $\mathscr{A}$ denotes the amplitude of gravity waves and the system of coordinates $(O, x, y, z)$ is chosen such that the $z$-axis points vertically upwards. Up to the numerical examples of § 4, we shall consider general shapes of the free surface: $\eta$ is only assumed to be a bounded regular function. The acoustic pressure field $P(M, t)$ defined at every point $M = (x, y, z)$ in $\Omega$ and every time $t$, must satisfy

$$\nabla^2 P - \frac{\partial^2 P}{\partial t^2} = 0 \quad \text{in} \quad \Omega, \tag{2.1a}$$

$$P = 0 \quad \text{on} \quad FS, \tag{2.1b}$$

$$\frac{\partial P}{\partial n} = F \quad \text{on} \quad \Gamma. \tag{2.1c}$$

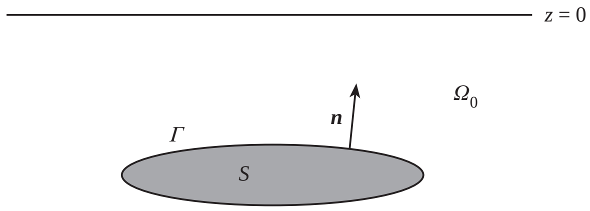


FIGURE 2. A body immersed in calm water.

Equation (2.1$a$) is the wave equation written here in a non-dimensioned form, (2.1$b$) expresses that pressure is constant above the free surface, and (2.1$c$) ensures the continuity of the normal velocity on the hull: $F$ is a datum which is related to the normal velocity $V_n$ of $\Gamma$ by $\partial P/\partial n = -\rho\, \partial V_n/\partial t$, where $\rho$ is the fluid density.

The direct solution of this problem seems to be very difficult, because of the corrugated free surface. However if we suppose that the amplitude $\mathscr{A}$ of the swell is small, we can use a perturbation method which consists in expanding the acoustic field according to ascending powers of this small parameter:

$$P = P_1 + \mathscr{A} P_2 + O(\mathscr{A}^2).$$

Substituting this expansion in the above equations leads us to write a sequence of problems set in the 'mean fluid domain' $\Omega_0$ bounded by 'the mean free surface' $FS_0$, namely the plane $z = 0$ (see figure 2). The first- and second-order acoustic pressures $P_1(M,t)$ and $P_2(M,t)$ are respectively solutions to

$$\nabla^2 P_1 - \frac{\partial^2 P_1}{\partial t^2} = 0 \quad \text{in} \quad \Omega_0, \tag{2.2$a$}$$

$$P_1 = 0 \quad \text{on} \quad FS_0, \tag{2.2$b$}$$

$$\frac{\partial P_1}{\partial n} = F \quad \text{on} \quad \Gamma, \tag{2.2$c$}$$

and

$$\nabla^2 P_2 - \frac{\partial^2 P_2}{\partial t^2} = 0 \quad \text{in} \quad \Omega_0, \tag{2.3$a$}$$

$$P_2 = Q \quad \text{on} \quad FS_0, \tag{2.3$b$}$$

$$\frac{\partial P_2}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{2.3$c$}$$

where

$$Q\,(M_0,t) = -\eta\,(M_0)\,\frac{\partial P_1}{\partial z}(M_0,t) \quad \text{for every} \quad M_0 \in FS_0. \tag{2.4}$$

Equations (2.2$b$) and (2.3$b$) follow from (2.1$b$) using a Taylor expansion of the acoustic pressure $P(x,y,\mathscr{A}\,\eta(x,y),t)$ about $z = 0$.

Suppose now that the body vibrates in a periodic and established way, which may be expressed by

$$F(M,t) = \mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t} f(M)\right\}, \quad M \in \Gamma,$$

where $\omega$ is the acoustic wave pulsation. This suggests seeking the first-order acoustic pressure in the form

$$P_1(M,t) = \mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t} p_1(M)\right\}.$$

Substituting this expression into (2.4), we deduce the same time-dependence for $Q$ and thus also for $P_2$:

$$P_2(M,t) = \mathrm{Im}\left\{ \mathrm{e}^{-\mathrm{i}\omega t} p_2(M) \right\}.$$

The fields $p_1$ and $p_2$ are thus solutions to the following first- and second-order time-harmonic problems:

$$\nabla^2 p_1 + \omega^2 p_1 = 0 \quad \text{in} \quad \Omega_0, \tag{2.5a}$$

$$p_1 = 0 \quad \text{on} \quad FS_0, \tag{2.5b}$$

$$\frac{\partial p_1}{\partial n} = f \quad \text{on} \quad \Gamma, \tag{2.5c}$$

and

$$\nabla^2 p_2 + \omega^2 p_2 = 0 \quad \text{in} \quad \Omega_0, \tag{2.6a}$$

$$p_2 = q \quad \text{on} \quad FS_0, \tag{2.6b}$$

$$\frac{\partial p_2}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{2.6c}$$

where

$$q = -\eta \left. \frac{\partial p_1}{\partial z} \right|_{FS_0}. \tag{2.7}$$

These systems of equations are incomplete for determining $p_1$ and $p_2$. Indeed there is no condition for their asymptotic behaviour at infinity; more precisely we cannot distinguish the *outgoing* waves from the *incoming* waves. And of course only the outgoing waves (i.e. which radiate towards infinity) are physically acceptable.

Concerning the first-order problem, we know how to write a condition which selects outgoing waves. For this kind of problem, where the condition on the free surface is homogeneous, this *radiation condition* is written in the Rellich form:

$$\frac{\partial p_1}{\partial R} - \mathrm{i}\omega p_1 = O\left(\frac{1}{R^2}\right) \quad \text{when} \quad R = \left(x^2 + y^2 + z^2\right)^{1/2} \to \infty. \tag{2.8}$$

On the other hand concerning the second-order problem, where the condition on the free surface extends up to infinity, we do not know how to write a condition at infinity to choose outgoing waves. It means that this problem is ill-posed: it has an infinity of solutions but it is likely that only one is physically acceptable.

In the rest of this article, we shall see how to get round this difficulty to find the physical solution of the second-order problem. In a first step, we shall consider the case where the body is substituted by a point source: it contains the difficulty but the calculations are explicit. We will judiciously select one solution and show why it is actually the physical solution.

## 3. Analytic solution for a time-harmonic point source

Let us consider a Helmholtz source located at point $A$ with coordinates $(0, 0, -a)$ for positive $a$ (see figure 3). In § 3.1, we shall assume the acoustic field generated by this source to be time-harmonic and formulate both first- and second-order problems. For the first-order problem, which is well-posed, we shall exhibit its unique solution. On the other hand, for the second-order problem we have no idea of what to impose at infinity to get a well-posed problem; we shall choose one solution via a Fourier

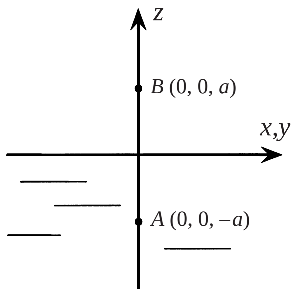


FIGURE 3. Acoustic point source.

transform. Then in § 3.2 we shall consider a time-harmonic source starting at $t = 0$ (generating a non-harmonic acoustic field) in the presence of gravity waves and solve the corresponding transient problem up to the second order. It will be shown that this solution tends for $t \to +\infty$ towards the previous time-harmonic solution of § 3.1, so establishing that it was in fact the right solution. This result is referred to as the *limiting amplitude principle*.

### 3.1. *An explicit solution of the time-harmonic problem*

#### 3.1.1. *First-order time-harmonic problem*

According to § 2, the first-order pressure generated by a time-harmonic point source is $\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t}\, p_1(M)\right\}$ where $p_1$ must satisfy

$$(\nabla^2 + \omega^2)p_1 = \delta_A \quad \text{for} \quad z < 0, \tag{3.1a}$$

$$p_1 = 0 \quad \text{on} \quad FS_0, \tag{3.1b}$$

$$\text{RC}. \tag{3.1c}$$

Here $\delta_A$ is the Dirac distribution at point $A$, and RC stands for radiation condition (2.8).

Such a problem is known to be well-posed. And its solution can be trivially constructed by the so-called *image procedure*, i.e. by superposing a Helmholtz source at point $A$ and a Helmholtz sink at point $B$, where $B$ is symmetrical to $A$ with respect to the mean free surface $FS_0$ (its coordinates are $(0, 0, a)$, see figure 3). Let $g$ denote the 'outgoing' Green function of the Helmholtz equation, i.e.

$$g(R) = \frac{\mathrm{e}^{\mathrm{i}\omega R}}{-4\pi R} \quad \text{with} \quad R = \left(x^2 + y^2 + z^2\right)^{1/2}, \tag{3.2}$$

which represents the field generated by a time-harmonic source in the free space (located at the origin). It may be readily seen that the solution to the above equations is

$$p_1(M) = g(\|AM\|) - g(\|BM\|), \tag{3.3}$$

where $\|AM\| = \left(x^2 + y^2 + (z + a)^2\right)^{1/2}$ and $\|BM\| = \left(x^2 + y^2 + (z - a)^2\right)^{1/2}$.

#### 3.1.2. *Second-order time-harmonic problem*

The second-order time-harmonic pressure field associated with the point source is $\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t} p_2(M)\right\}$ where $p_2$ must satisfy

$$(\nabla^2 + \omega^2)p_2 = 0 \quad \text{for} \quad z < 0, \tag{3.4a}$$

$$p_2 = q \quad \text{on} \quad FS_0, \tag{3.4b}$$

and the right-hand side of (3.4*b*) can be derived from $p_1$ (see (2.7)):

$$q(M_0) = -\eta(M_0) \quad \frac{\partial p_1}{\partial z}(M_0) \quad \text{for every} \quad M_0 \in FS_0. \tag{3.5}$$

As $p_2$ is clearly a smooth function and as the calculation domain is geometrically simple we shall try to solve (3.4*a, b*) by using a horizontal Fourier transform. Namely let us denote $u$ and $v$ the transformed variables corresponding to $x$ and $y$, and

$$\widehat{p_2}(u,v,z) = \mathscr{F}_{x,y}\, p_2(x,y,z) = \iint_{\mathbb{R}^2} p_2(x,y,z)\, \mathrm{e}^{-2\mathrm{i}\pi(ux+vy)}\, \mathrm{d}x\, \mathrm{d}y.$$

Assuming that the partial $z$-derivative commutes with $\mathscr{F}$, we see that $\widehat{p_2}$ must satisfy

$$\frac{\partial^2 \widehat{p_2}}{\partial z^2} + \left(-4\pi^2\varrho^2 + \omega^2\right)\widehat{p_2} = 0 \quad \text{for} \quad z < 0, \tag{3.6a}$$

$$\widehat{p_2} = \widehat{q} \quad \text{for} \quad z = 0, \tag{3.6b}$$

where $\varrho = \sqrt{u^2+v^2}$. The general solution of (3.6*a*) can be written

$$A(u,v)\, \mathrm{e}^{-\mathrm{i}z(\omega^2-4\pi^2\varrho^2)^{1/2}} + B(u,v)\, \mathrm{e}^{+\mathrm{i}z(\omega^2-4\pi^2\varrho^2)^{1/2}} \quad \text{for} \quad 0 < \varrho < \frac{\omega}{2\pi},$$

$$C(u,v)\, \mathrm{e}^{z(4\pi^2\varrho^2-\omega^2)^{1/2}} + D(u,v)\, \mathrm{e}^{-z(4\pi^2\varrho^2-\omega^2)^{1/2}} \quad \text{for} \quad \varrho > \frac{\omega}{2\pi}.$$

That $\widehat{p_2}$ should be bounded for $z \leqslant 0$ implies $D = 0$.

This expression can be written in a more convenient form using the ‘outgoing’ Green function $g(R)$ given in (3.2) as well as the ‘incoming’ Green function $g^*(R) = \overline{g(R)}$. More precisely, consider the associated doublets $\partial g/\partial z$ and $\partial g^*/\partial z$ in the $z$-direction. Using polar coordinates and introducing $\mathrm{J}_0$, taking into account $\mathrm{J}_0' = -J_1$ and integrating by parts, and finally using Erdélyi *et al.* (1954, p. 20, formulae 18 and 21) it is easy to find their Fourier transforms, namely

$$\widehat{\frac{\partial g}{\partial z}}(\varrho,z) = \begin{cases} -\frac{1}{2}\,\mathrm{e}^{-\mathrm{i}z(\omega^2-4\pi^2\varrho^2)^{1/2}} & \text{for} \quad 0 < \varrho < \dfrac{\omega}{2\pi}, \\ -\frac{1}{2}\,\mathrm{e}^{z(\omega^2-4\pi^2\varrho^2)^{1/2}} & \text{for} \quad \varrho > \dfrac{\omega}{2\pi}, \end{cases} \tag{3.7a}$$

$$\widehat{\frac{\partial g^*}{\partial z}}(\varrho,z) = \begin{cases} -\frac{1}{2}\,\mathrm{e}^{+\mathrm{i}z(\omega^2-4\pi^2\varrho^2)^{1/2}} & \text{for} \quad 0 \leqslant \varrho < \dfrac{\omega}{2\pi}, \\ -\frac{1}{2}\,\mathrm{e}^{z(4\pi^2\varrho^2-\omega^2)^{1/2}} & \text{for} \quad \varrho > \dfrac{\omega}{2\pi}. \end{cases} \tag{3.7b}$$

Let us arbitrarily split $C$ into two parts: $C(u,v) = E(u,v) + F(u,v)$. And let $\mu(x,y)$ and $\mu^*(x,y)$ be the respective inverse Fourier transforms of

$$\widehat{\mu}(u,v) = \begin{cases} A & \text{for} \quad 0 \leqslant \varrho \leqslant \dfrac{\omega}{2\pi}, \\ E & \text{for} \quad \varrho > \dfrac{\omega}{2\pi}, \end{cases} \quad \text{and} \quad \widehat{\mu^*}(u,v) = \begin{cases} B & \text{for} \quad 0 \leqslant \varrho \leqslant \dfrac{\omega}{2\pi}, \\ F & \text{for} \quad \varrho > \dfrac{\omega}{2\pi}. \end{cases}$$

Then we have

$$\widehat{p_2}(u,v,z) = -2\,\widehat{\mu}(u,v)\,\widehat{\frac{\partial g}{\partial z}}(\varrho,z) - 2\,\widehat{\mu^*}(u,v)\,\widehat{\frac{\partial g^*}{\partial z}}(\varrho,z) \tag{3.8}$$

for $z \leqslant 0$. Equation (3.6*b*) together with (3.7*a*) implies $\widehat{\mu} + \widehat{\mu^*} = \widehat{q}$, i.e.

$$\mu(x,y) + \mu^*(x,y) = q(x,y). \tag{3.9}$$

Via $\mathscr{F}^{-1}$, (3.8) becomes in a quite formal way

$$p_2(x,y,z) = -2\,\mu(x,y) \overset{x,y}{*} \frac{\partial g}{\partial z}(x,y,z) - 2\,\mu^*(x,y) \overset{x,y}{*} \frac{\partial g^*}{\partial z}(x,y,z), \tag{3.10}$$

$\mu$ and $\mu^*$ being compelled to satisfy (3.9) and $\overset{x,y}{*}$ meaning the convolution with respect to $x$ and $y$.

Here we decide to make a fundamental choice. The Rellich Condition in the first-order problem allows us to select outgoing waves (see § 2); therefore, we also select the outgoing waves in the second-order problem assuming $\mu^* = 0$, whence $\mu = q$ and

$$p_2(M) = \left(-2\,q \overset{x,y}{*} \frac{\partial g}{\partial z}\right)(M) = -2\iint_{FS_0} q(M_0)\,\frac{\partial}{\partial z} g(\|M_0M\|)\,\mathrm{d}M_0. \tag{3.11}$$

¿From (3.3) and (3.5) we see that for every $M_0 \in FS_0$,

$$q\,(M_0) = \frac{a}{2\pi}\,\eta(M_0)\left(\frac{\mathrm{i}\omega}{\|AM_0\|^2} - \frac{1}{\|AM_0\|^3}\right)\mathrm{e}^{\mathrm{i}\omega\|AM_0\|} \tag{3.12}$$

(with $\|AM_0\| = \left(x^2+y^2+a^2\right)^{1/2}$), while

$$\frac{\partial}{\partial z} g(\|M_0M\|) = -\frac{z}{4\pi}\left(\frac{\mathrm{i}\omega}{\|M_0M\|^2} - \frac{1}{\|M_0M\|^3}\right)\mathrm{e}^{\mathrm{i}\omega\|M_0M\|}. \tag{3.13}$$

Hence, for any given negative $z$, the convolution integral (3.11) is absolutely convergent. It defines a $C^\infty$ function which is actually a solution of (3.4*a*). Moreover it satisfies the Dirichlet condition (3.4*b*) since it is well-known that a smooth double-layer potential

$$\mathscr{D}(M) = \int_\Sigma \mu(N)\,\frac{\partial}{\partial n_N} g(\|NM\|)\,\mathrm{d}\Sigma_N$$

on a surface $\Sigma$ exhibits the limits

$$\mp\frac{\mu(M_0)}{2} + \int_\Sigma \mu(N)\,\frac{\partial}{\partial n_N} g(\|NM_0\|)\,\mathrm{d}\Sigma_N \tag{3.14}$$

when $M$ tends towards $M_0 \in \Sigma$, with the + sign if the limit is obtained on the side of $\Sigma$ containing the normal vector $n$, and the − sign on the opposite side. If $\Sigma$ is a plane, the integral in (3.14) vanishes and the Dirichlet condition (3.4*b*) is trivially satisfied (note that $\partial/\partial n_N = -\partial/\partial z$).

But if we replace $g$ by $g^*$ in (3.11) the same conclusions are valid. Still better: if we split $q$, according to (3.9), into two continuous functions decreasing at infinity the same conclusions hold. So it must be pointed out that we have chosen one solution among an infinity. Now by using a *limiting amplitude procedure* we shall prove that we have chosen the right solution — we mean the physical one.

There remains an open question as regards this choice: can we formulate a condition at infinity which allows us to select this physical solution. An attempt was made by Euvrard (1994*b*) to decide whether the standard radiation condition (2.8) fulfils the purpose or not. This requires us to exhibit the asymptotic behaviour of $p_2$ at infinity. Unfortunately, a mistake in the calculations does not allow us to validate the conclusions of this paper.

## 3.2. *Justification of the solution*

### 3.2.1. *First-order time-dependent problem*

Consider now a time-harmonic source starting at $t = 0$. Without gravity waves, the pressure field generated by this source is solution to the time-dependent first-order problem

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) P_1 = \delta_A \otimes \mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t}\right\} \quad \text{for} \quad z < 0 \quad \text{and} \quad t > 0, \tag{3.15$a$}$$

$$P_1 = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad t > 0, \tag{3.15$b$}$$

$$P_1 = 0 \quad \text{and} \quad \frac{\partial P_1}{\partial t} = 0 \quad \text{for} \quad z < 0 \quad \text{and} \quad t = 0. \tag{3.15$c$}$$

The solution to these equations can be expressed by means of the Green function $G$ of the wave equation in the free space: $G$ is a distribution whose support is exactly the boundary of the *forward light cone* (i.e. the set $\{(M,t);\ t > 0 \text{ and } \|OM\| = t\}$, see Treves 1975); it is defined by:

$$G(M,t) = -\frac{1}{4\pi t}\,\delta(\|OM\| - t), \tag{3.16}$$

which satisfies

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) G = \delta_{M=O} \otimes \delta_{t=0}.$$

The latter allows us to express the solution to the wave equation

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) P = T, \tag{3.17}$$

written in the sense of distributions in $\mathbb{R}^4$: here $P$ and $T$ are distributions defined in the whole space $\mathbb{R}^3$ and for all $t \in \mathbb{R}$; they are both assumed to be causal, i.e. $P = 0$ and $T = 0$ if $t < 0$. In this situation, the convolution product

$$T \overset{x,y,z,t}{*} G$$

is actually the only causal solution to (3.17). Indeed

$$\begin{aligned}\left\{\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) P\right\} \overset{x,y,z,t}{*} G &= P \overset{x,y,z,t}{*} \left\{\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) G\right\} \\ &= P \overset{x,y,z,t}{*} \{\delta_{M=O} \otimes \delta_{t=0}\} \\ &= P.\end{aligned}$$

To find $P_1$, we have to rewrite (3.15*a–c*) in the form (3.17). We again use the *image procedure* which amounts to superimposing a source at point $A$ and a sink at point $B$ (see figure 3) in order to 'eliminate' the condition on $z = 0$. Consider the field (again denoted by $P_1$) defined in the whole space $\mathbb{R}^3$ by an antisymmetrical extension across the plane $z = 0$ (i.e. $P_1(x,y,z,t) = -P_1(x,y,-z,t)$ if $z > 0$), and also defined for negative $t$ by assuming that $P_1 = 0$ if $t < 0$. Using (3.15*a–c*), it is easily seen (see e.g. Schwartz 1965) that

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) P_1 = (\delta_A - \delta_B) \otimes \mathrm{Im}\left\{\mathrm{H}(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\},$$

in the sense of distributions (in $\mathbb{R}^4$). Here, $\mathrm{H}(t)$ stands for the Heaviside step function.

We deduce that

$$P_1 = \big((\delta_A - \delta_B) \otimes \mathrm{Im}\left\{\mathrm{H}(t)\ \mathrm{e}^{-\mathrm{i}\omega t}\right\}\big) \overset{x,y,z,t}{*} G,$$

which yields

$$P_1(M,t) = \mathrm{Im}\left\{\mathrm{H}(t-\|AM\|)\ \frac{\mathrm{e}^{-\mathrm{i}\omega(t-\|AM\|)}}{-4\pi\|AM\|} - \mathrm{H}(t-\|BM\|)\ \frac{\mathrm{e}^{-\mathrm{i}\omega(t-\|BM\|)}}{-4\pi\|BM\|}\right\}, \tag{3.18}$$

by virtue of the expression (3.16) for $G$.

The latter expression can be compared with the time-harmonic solution $p_1$ given in (3.3): we clearly see that

$$P_1(M,t) = \mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t}\ p_1(M)\right\} \quad \text{if} \quad t > \max\left\{\|AM\|, \|BM\|\right\}. \tag{3.19}$$

In other words, at every given point $M$, the time-dependent solution coincides for large enough $t$ with the time-harmonic solution: this may be seen as a simplified form of the *limiting amplitude principle* for the first-order pressure field.

Our aim is now to prove a similar result for the second-order problem. We will see that the second-order pressure field is no longer time-harmonic but tends asymptotically to the time-harmonic field exhibited in § 3.1.2.

3.2.2. *Second-order time-dependent problem*

The time-dependent second-order problem for the source starting at $t = 0$ is

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) P_2 = 0 \quad \text{for} \quad z < 0 \quad \text{and} \quad t > 0, \tag{3.20$a$}$$

$$P_2 = Q \quad \text{for} \quad z = 0 \quad \text{and} \quad t > 0, \tag{3.20$b$}$$

$$P_2 = 0 \quad \text{and} \quad \frac{\partial P_2}{\partial t} = 0 \quad \text{for} \quad z < 0 \quad \text{and} \quad t = 0, \tag{3.20$c$}$$

where the transient Dirichlet datum is $Q = -\eta\ \partial P_1/\partial z_{|z=0}$ (see (2.4)). From (3.18), we deduce that for every $M_0 \in FS_0$,

$$Q\,(M_0,t) = \mathrm{H}\left(t - \|AM_0\|\right)\ K\,(M_0,t), \tag{3.21}$$

where

$$K\,(M_0,t) = \frac{a}{2\pi}\ \eta\,(M_0)\ \ \mathrm{Im}\left\{\left(\frac{\mathrm{i}\omega}{\|AM_0\|^2} - \frac{1}{\|AM_0\|^3}\right)\mathrm{e}^{\mathrm{i}\omega\left(\|AM_0\|-t\right)}\right\}.$$

Note that by virtue of (3.12), we have

$$Q\,(M_0,t) = \mathrm{H}\left(t - \|AM_0\|\right)\ \ \mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\omega t}\ q(M_0)\right\}. \tag{3.22}$$

In order to find $P_2$, we apply the image procedure again. The difference with the first-order problem is that the antisymmetrical extension of $P_2$ is no longer continuous across the plane $z = 0$. Using the standard distribution theory (see Schwartz 1965), we deduce in this case that (3.20*a*–*c*) is equivalent to

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) P_2 = -2\frac{\partial}{\partial z}\left(Q\ \delta_{FS_0}\right), \tag{3.23}$$

in the sense of distributions (in $\mathbb{R}^4$). Here $\delta_{FS_0}$ denotes the surface Dirac measure on the plane $z = 0$. Hence, as for the first-order problem, we infer that

$$P_2 = \left\{-2\frac{\partial}{\partial z}\left(Q\ \delta_{FS_0}\right)\right\} \overset{x,y,z,t}{*} G,$$

which yields

$$P_2 = -2\frac{\partial}{\partial z}(Q \overset{x,y,t}{*} G) = -2Q \overset{x,y,t}{*} \frac{\partial G}{\partial z}. \tag{3.24}$$

3.2.3. *The limiting-amplitude procedure*

To justify our choice of a solution of the second-order time-harmonic problem (see (3.11)), we prove here that it actually is the asymptotic behaviour of the time-dependent pressure field $P_2$ as $t \to +\infty$. More precisely we shall see that

$$P_2(M,t) = \operatorname{Im}\left\{e^{-i\omega t}\, p_2(M)\right\} + O(t^{-2}) \quad \text{as} \quad t \to +\infty, \tag{3.25}$$

for every fixed $M$: this result is referred to as the *limiting-amplitude principle.*

First we need a more explicit expression for $P_2$ given in (3.24). ¿From the definition (3.16) of $G$, we infer

$$(Q \overset{x,y,t}{*} G)(M,t) = \iint_{FS_0} \frac{Q\left(M_0, t - \|M_0M\|\right)}{-4\pi\,\|M_0M\|} \mathrm{d}M_0.$$

Notice that for fixed $M$ and $t$, the domain of integration is bounded. Indeed, we deduce from the expression (3.22) for $Q$ that

$$(Q \overset{x,y,t}{*} G)(M,t) = \operatorname{Im} \iint_{FS_0} \frac{\mathrm{H}\left(t - \|M_0M\| - \|AM_0\|\right)\; q(M_0)\; e^{-i\omega\left(t-\|M_0M\|\right)}}{-4\pi\,\|M_0M\|} \mathrm{d}M_0,$$

which can be written as

$$(Q \overset{x,y,t}{*} G)(M,t) = \operatorname{Im} \iint_{\mathscr{D}(M,t)} \frac{q(M_0)\; e^{-i\omega\left(t-\|M_0M\|\right)}}{-4\pi\,\|M_0M\|} \mathrm{d}M_0,$$

where $\mathscr{D}(M,t)$ denotes the bounded subdomain of the mean free surface $FS_0$ defined by

$$\mathscr{D}(M,t) = \left\{M_0 \in FS_0;\; \|M_0M\| + \|AM_0\| < t\right\}.$$

Finally, using the expression for the Helmholtz Green function $g$ (see (3.2)), we have

$$P_2(M,t) = \operatorname{Im}\left\{-2\, e^{-i\omega t}\, \frac{\partial}{\partial z} \iint_{\mathscr{D}(M,t)} q(M_0)\; g(\|M_0M\|)\,\mathrm{d}M_0\right\}. \tag{3.26}$$

In order to apply the operator $\partial/\partial z$ to the double integral, we have to take into account the $z$-dependence of $\mathscr{D}(M,t)$ and its boundary $\partial\mathscr{D}(M,t)$. We shall actually consider $z$ as a 'time-like coordinate' and the integration domain $\mathscr{D}(M,t)$ as a 'material domain' to be followed in its movement, namely

$$\begin{aligned} &\frac{\partial}{\partial z} \iint_{\mathscr{D}(M,t)} \mathscr{I}(M,M_0)\,\mathrm{d}M_0 \\ &\qquad = \iint_{\mathscr{D}(M,t)} \frac{\partial \mathscr{I}}{\partial z}(M,M_0)\,\mathrm{d}M_0 + \int_{\partial\mathscr{D}(M,t)} \mathscr{I}(M,M_0)\, V_n\,\mathrm{d}s(M_0), \end{aligned} \tag{3.27}$$

where $\mathscr{I}(M,M_0) = q(M_0)\; g\left(\|M_0M\|\right)$ and $V_n$ is the 'normal velocity' of $\partial\mathscr{D}(M,t)$, i.e.

$$V_n = \frac{-\partial\mathscr{F}/\partial z(M,M_0)}{\|\nabla_{M_0}\mathscr{F}(M,M_0)\|} \quad \text{with} \quad \mathscr{F}(M,M_0) = \|M_0M\| + \|AM_0\| = t.$$

It remains to evaluate the asymptotic behaviour when $t \to +\infty$ of both integrals on the right-hand side of (3.27). Let us give a geometrical interpretation of the domains of integration. For fixed $M$ and $t$, the equation $\mathscr{F}(M, M_0) = t$ defines an ellipsoid of revolution with major axis $t$ and foci $A$ and $M$: for large $t$, it tends to a sphere of diameter $t$. Hence, $\partial\mathscr{D}(M,t)$ is the intersection of this ellipsoid with the free surface $FS_0$: it behaves like a circle of diameter $t$ when $t \to +\infty$. As a consequence

$$\int_{\partial\mathscr{D}(M,t)} \mathscr{I}(M, M_0)\, V_n \mathrm{d}s = O(t^{-3}),$$

since $\mathscr{I}(M, M_0) = O(t^{-3})$ and $V_n = O(t^{-1})$ if $M_0 \in \partial\mathscr{D}(M,t)$. On the other hand, the surface integral on $\mathscr{D}(M,t)$ tends to the corresponding integral on the whole plane $FS_0$. More precisely,

$$P_2(M,t) = \mathrm{Im}\left\{-2\, \mathrm{e}^{-\mathrm{i}\omega t} \iint_{FS_0} q(M_0)\, \frac{\partial}{\partial z} g\left(\|M_0 M\|\right) \mathrm{d}M_0\right\} + O(t^{-2}), \tag{3.28}$$

since $\partial\mathscr{I}/\partial z(M, M_0)$ is of order $\|OM_0\|^{-4}$ when $\|OM_0\| \to \infty$. By virtue of (3.11), this completes the proof of the form (3.25) of the limiting-amplitude principle.

## 4. Numerical solution for an immersed vibrating body

Let us come back to the general problem of an immersed body $S$ vibrating in a given time-harmonic way in the presence of gravity waves, as presented in § 2. The first-order complex pressure $p_1$ must satisfy

$$\nabla^2 p_1 + \omega^2 p_1 = 0 \quad \text{in} \quad \Omega_0, \tag{4.1$a$}$$

$$p_1 = 0 \quad \text{on} \quad FS_0, \tag{4.1$b$}$$

$$\frac{\partial p_1}{\partial n} = f \quad \text{on} \quad \Gamma, \tag{4.1$c$}$$

$$\text{RC}, \tag{4.1$d$}$$

where RC stands for the standard radiation condition (2.8). This problem is well-posed: we show below how to obtain a numerical approximation of $p_1$.

The second-order complex pressure $p_2$ must satisfy

$$\nabla^2 p_2 + \omega^2 p_2 = 0 \quad \text{in} \quad \Omega_0, \tag{4.2$a$}$$

$$p_2 = q \quad \text{on} \quad FS_0, \tag{4.2$b$}$$

$$\frac{\partial p_2}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{4.2$c$}$$

where $q = -\eta\ \partial p_1/\partial z_{|FS_0}$ (see (2.7)). For our numerical simulation, we shall assume that the shape of the corrugated free surface $\eta$ corresponds to a monochromatic plane swell, for instance

$$\eta(x,y) = \cos(\nu y). \tag{4.3}$$

This assumption is valid if the body is deeply immersed since we can neglect the effect of the body on the propagation of the swell near the free surface.

We mentioned in § 2 our lack of knowledge about what to impose at infinity so that the above second-order problem should be well-posed because of the presence of the non-homogeneous boundary condition $p_2 = q$ on the unbounded surface $FS_0$. The aim of this section is to show how to overcome this difficulty: we propose to split problem (4.2$a$–$c$) into two successive problems via the linearity of the equations.

4.1. *A two-step procedure for the second-order problem*

Let us first ignore the presence of the body $S$ and consider the following problem for the 'free' second-order pressure field $p_2^f$:

$$\nabla^2 p_2^f + \omega^2 p_2^f = 0 \quad \text{in the whole half-space} \quad z < 0, \tag{4.4a}$$

$$p_2^f = q \quad \text{on} \quad FS_0. \tag{4.4b}$$

Of course condition (4.2$c$) is not satisfied by $p_2^f$. So $p_2^f$ will be corrected by a second pressure field $p_2^c$ satisfying

$$\nabla^2 p_2^c + \omega^2 p_2^c = 0 \quad \text{in} \quad \Omega_0, \tag{4.5a}$$

$$p_2^c = 0 \quad \text{on} \quad FS_0, \tag{4.5b}$$

$$\frac{\partial p_2^c}{\partial n} = -\frac{\partial p_2^f}{\partial n} \quad \text{on} \quad \Gamma, \tag{4.5c}$$

$$\text{RC}. \tag{4.5d}$$

The sum $p_2 = p_2^f + p_2^c$ obviously satisfies (4.2$a$–$c$). As previously explained the radiation condition is associated here with the homogeneous Dirichlet condition on $FS_0$.

The general concept of spliting $p_2$ into $p_2^f$ defined in the half-space $z < 0$ and $p_2^c$ in $\Omega_0$ ($p_2^c$ satisfing RC but not $p_2^f$), is not new since it has been used for a long time in all diffraction problems: the incident potential ignores the body and does not satisfy RC, while the diffracted potential is defined outside the body and satisfies a Neumann condition such as (4.5$c$), together with RC.

But the question is: how to solve (4.4$a,b$)? Due to the very special geometry of the domain where this problem is posed, it is possible to apply a horizontal Fourier transform exactly like in § 3.1: it actually is the same problem as (3.4$a,b$), the only difference being that now $q$ is no longer explicit but is derived from the numerical approximation of $p_1$. The method presented in § 3.1 remains valid; we shall make the same choice $\mu^* = 0$, and $p_2^f$ will be the convolution product

$$p_2^f(M) = -2 \iint_{FS_0} q(M_0) \frac{\partial}{\partial z} g(\| M_0 M \|) \, \mathrm{d}M_0, \tag{4.6}$$

where the vertical Helmholtz doublet $\partial g / \partial z$ is given in (3.13).

Is our choice of a condition at infinity, which is implicit in (4.6), the right one? There is no simple proof for that, but good presumptive evidence. The only way to justify this choice would be to use a *limiting amplitude process* as we did for a point source. But the fact that the solution can no longer be expressed explicitly makes the justification far more difficult: a sketch of the proof will be mentioned in § 5.

4.2. *A FEM/BEM coupling*

To solve problems (4.1$a$–$d$) and (4.5$a$–$d$) we used an efficient method which was designed by Jami & Lenoir (1978) and seems to be well adapted here. It couples the boundary element method (BEM) and the finite element method (FEM): it will be briefly reported hereafter for the example of the first-order problem.

The starting point of this method is the well-known Green representation formula

$$p_1(M) = \int_\Gamma \left[ -p_1(N) \frac{\partial \mathcal{G}}{\partial n_N}(M,N) + f(N) \, \mathcal{G}(M,N) \right] \mathrm{d}\Gamma_N \quad \text{for} \quad M \in \Omega_0, \tag{4.7}$$

where $\mathcal{G}$ denotes the Green function of the first-order problem, i.e. the time-harmonic

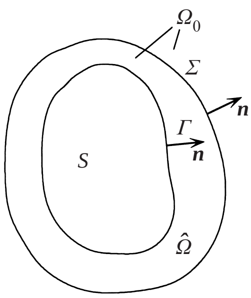


FIGURE 4. A BEM/FEM coupling.

field generated by a source located at point $N$ (up to a suitable translation, it is simply the field calculated in § 3.1):

$$\mathscr{G}(M,N) = g(\|MN\|) - g(\|MN'\|) \quad \text{where} \quad g(R) = \frac{\mathrm{e}^{\mathrm{i}\omega R}}{-4\pi R},$$

$N'$ being symmetrical with $N$ with respect to the plane $z = 0$.

The coupling method consists in considering instead of the unbounded domain $\Omega_0$, a bounded domain $\hat{\Omega}$ limited by the surfaces $\Sigma$ and $\Gamma$ (see figure 4), where $\Sigma$ is any smooth surface surrounding $\Gamma$; it may be located close to $\Gamma$ but is compelled to be at a strictly positive distance from $\Gamma$. In $\hat{\Omega}$ and on $\Gamma$ the Helmholtz equation (4.1$a$) and the Neumann condition (4.1$c$) are imposed. On $\Sigma$ the following condition, called *coupling condition* is imposed:

$$\mathrm{D}p_1(M) = \mathrm{D}\left\{\int_\Gamma \left[-p_1(N)\,\frac{\partial\mathscr{G}}{\partial n_N}(M,N) + f(N)\,\mathscr{G}(M,N)\right]\mathrm{d}\Gamma_N\right\} \quad \forall M \in \Sigma, \tag{4.8}$$

which simply derives from (4.7) by applying the boundary operator

$$\mathrm{D} = \frac{\partial}{\partial n_M} + \lambda \quad \text{for some complex constant } \lambda \text{ such that} \quad \operatorname{Im}\lambda \neq 0.$$

This condition actually 'couples' the unknown $p_1$ on $\Sigma$ and the unknown $p_1$ on $\Gamma$.

The problem to be solved in $\hat{\Omega}$ can be written as

$$\nabla^2 p_1 + \omega^2 p_1 = 0 \quad \text{in} \quad \hat{\Omega}, \tag{4.9$a$}$$

$$\frac{\partial p_1}{\partial n} = f \quad \text{on} \quad \Gamma, \tag{4.9$b$}$$

$$\text{coupling condition} \quad \text{on} \quad \Sigma. \tag{4.9$c$}$$

The reason for introducing the operator D is the elimination of the so-called *irregular frequencies*. Indeed, it can be proved (see Euvrard 1994$a$) that (4.9$a$–$c$) is equivalent (before discretization) to (4.1$a$–$d$) provided the 'auxiliary problem'

$$\nabla^2 p + \omega^2 p = 0 \quad \text{in} \quad \bar{S} \cup \hat{\Omega},$$
$$\mathrm{D}p = 0 \quad \text{on} \quad \Sigma,$$

only has the trivial solution $p = 0$: and this property clearly holds if $\operatorname{Im}\lambda \neq 0$. In this case, this means that the solution to (4.9$a$–$c$) is the restriction to $\hat{\Omega}$ of the solution to (4.1$a$–$d$), and conversely that the solution to (4.9$a$–$c$) can be analytically extended to $\Omega_0$ into the solution to (4.1$a$–$d$). Such an extension is quite easy: it is simply the representation formula (4.7).

Then problem (4.9*a–c*) is written in a variational form and discretized using finite elements. After resolution of the subsequent linear algebraic system, the solution is given in $\hat{\Omega}$ and on $\Gamma$ via the finite element interpolation; but it is also given everywhere in $\Omega_0$ via the integral representation (4.7), involving a numerical quadrature of the integral on $\Gamma$. Unlike conventional BEM, no singular function appears in this method since $\mathscr{G}(M,N)$ is computed for $M \in \Sigma$ and $N \in \Gamma$, the distance $\|MN\|$ being strictly positive; this allows the introduction of finite elements of high order and of numerical quadrature in (4.7). A code devoted to this method has been developed by our team in the last years: it is called MELINA and has been used to obtain the numerical results given hereafter in § 4.4.

### 4.3. *Numerical computation of the convolution product*

The aim of this subsection is to show a way to compute the convolution product (4.6) which defines the 'free' second-order pressure field $p_2^f$. As $p_1$ is only known by the numerical process described above, the same holds for the datum $q = -\eta\, \partial p_1/\partial z_{|FS_0}$ (see (2.7)). The method proposed here consists in a numerical integration of (4.6) based on Fourier transform. Compared with a direct computation of (4.6), this method becomes less time consuming if $p_2^f$ is determined at a large number of points, by using efficient discrete Fourier algorithms.

The convolution product (4.6) can be expressed by means of the horizontal Fourier transform (with respect to $x$ and $y$):

$$p_2^f = -2\, \mathscr{F}^{-1}\left(\mathscr{F}(q) \cdot \mathscr{F}\left(\frac{\partial g}{\partial z}\right)\right), \tag{4.10}$$

where $\mathscr{F}(\partial g/\partial z)$ is explicit (see (3.7*a*)). The main difficulty lies in the calculation of $\mathscr{F}(q)$: indeed $q$ behaves like $O((x^2+y^2)^{-1})$ at infinity, which is not absolutely integrable ($q \notin L^1(FS_0)$). This prevents us from using a standard fast fourier transform process. Our idea consists in exhibiting the asymptotic expansion $q_{as}$ of $q$ at infinity, in order to split $\mathscr{F}(q)$ in two parts:

$$\mathscr{F}(q) = \mathscr{F}\,(q - q_{as}) + \mathscr{F}\,(q_{as}), \tag{4.11}$$

where $q - q_{as} \in L^1(FS_0)$, which allows us to calculate $\mathscr{F}(q - q_{as})$ thanks to a fast Fourier transform algorithm, and $\mathscr{F}(q_{as})$ involves explicit calculations.

The asymptotic behaviour of $q$ is easily derived from the integral representation formula (4.7) (using the asymptotic expansion of $\mathscr{G}$ and its derivatives). In polar coordinates, we obtain

$$q_{as}(r,\theta) = \eta(r\,\sin\theta)\; \tilde{q}_{as}(r,\theta) \quad \text{where} \quad \tilde{q}_{as}(r,\theta) = \chi(r)\, \frac{e^{i\omega r}}{r^2}\, h(\theta), \tag{4.12}$$

and $h(\theta)$ is defined by the following integral on the hull $\Gamma$ of the body:

$$h(\theta) = -\frac{\omega}{2\pi}\int_\Gamma (p_1(N)\,\alpha_\theta(N)\cdot n - \mathrm{i} z_N\, f(N))\, e^{-\mathrm{i}\omega(x_N\cos\theta + y_N \sin\theta)}\, d\Gamma_N, \tag{4.13}$$

with $\alpha_\theta(N) = (\omega\, z_N \cos\theta,\, \omega\, z_N \sin\theta,\, \mathrm{i})$; the radial function $\chi(r)$ is a given regular truncation function which avoids the singularity of $r^{-2}$ at $r = 0$: for some fixed $a$ and $b$ such that $0 < a < b$,

$$\chi(r) = 0 \quad \text{if} \quad r < a \quad \text{and} \quad \chi(r) = 1 \quad \text{if} \quad r > b.$$

It is readily seen that the Fourier transform of $q_{as}$ must be taken in an $L^2$ sense.

To calculate this Fourier transform, we first rewrite $\eta$ (given in (4.3)) in complex exponential form, which yields

$$\mathscr{F}\, q_{as}(u,v) = \frac{1}{2}\left\{\mathscr{F}\, \tilde{q}_{as}\left(u, v - \frac{v}{2\pi}\right) + \mathscr{F}\, \tilde{q}_{as}\left(u, v + \frac{v}{2\pi}\right)\right\}.$$

We thus have to compute $\mathscr{F}\, \tilde{q}_{as}$. Noticing that $\tilde{q}_{as}$ is the product of a function of $r$ by $h(\theta)$, we can express its Fourier transform by converting $h(\theta)$ in Fourier series:

$$h(\theta) = \sum_{n=-\infty}^{+\infty} \hat{h}_n\, \mathrm{e}^{\mathrm{i}n\theta} \quad \text{where} \quad \hat{h}_n = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, \mathrm{e}^{-\mathrm{i}n\theta}\, \mathrm{d}\theta.$$

If we denote $(\rho, \alpha)$ the polar coordinates in the Fourier plane, we have

$$\mathscr{F}\, \tilde{q}_{as}(\rho, \alpha) = \sum_{n=-\infty}^{+\infty} (-\mathrm{i})^n\, \mathrm{e}^{\mathrm{i}n\alpha}\, \hat{h}_n\, \mathscr{L}_n(\rho)$$

where

$$\mathscr{L}_n(\rho) = 2\pi \int_0^{+\infty} \chi(r)\, \frac{\mathrm{e}^{\mathrm{i}\omega r}}{r}\, J_n(2\pi\rho r)\, \mathrm{d}r.$$

For the computation of $\mathscr{L}_n(\rho)$, we have to distinguish the two cases $n \neq 0$ and $n = 0$.

If $n \neq 0$, by virtue of the choice of the truncation function $\chi(r)$, we can rewrite $\mathscr{L}_n(\rho)$ in the form $\mathscr{L}_n(\rho) = \mathscr{J}_n(\rho) + \mathscr{K}_n(\rho)$ where

$$\mathscr{J}_n(\rho) = 2\pi \int_0^{b} (1 - \chi(r))\, \frac{\mathrm{e}^{\mathrm{i}\omega r}}{r}\, J_n(2\pi\rho r)\, \mathrm{d}r \quad \text{and} \quad \mathscr{K}_n(\rho) = 2\pi \int_0^{+\infty} \frac{\mathrm{e}^{\mathrm{i}\omega r}}{r}\, J_n(2\pi\rho r)\, \mathrm{d}r.$$

The expression for $\mathscr{K}_n(\rho)$ is explicit (see Gradshteyn & Ryzhik 1980). If $n > 0$, we have

$$\mathscr{K}_n(\rho) = \begin{cases} 2\pi\, \dfrac{\mathrm{e}^{\mathrm{i}n\pi/2}}{n} \left(\dfrac{2\pi\rho/\omega}{1 + \left(1 - (2\pi\rho/\omega)^2\right)^{1/2}}\right)^n & \text{if} \quad \rho < \omega/2\pi, \\ 2\pi\, \dfrac{\mathrm{e}^{\mathrm{i}n \arcsin(\omega/(2\pi\rho))}}{n} & \text{if} \quad \rho > \omega/2\pi, \end{cases}$$

and $\mathscr{K}_{-n}(\rho) = (-1)^n \mathscr{K}_n(\rho)$. On the other hand, $\mathscr{J}_n(\rho)$ is computed by a numerical integration. In order to take into account the oscillations due to the complex exponential and the Bessel function of first kind, we decompose the integration domain $[0, b]$ into subdomains $[a_i, a_{i+1}]$ where the $a_i$ are the zeros of the integrand except perhaps at 0 and $b$. In each range of integration, we substitute a series of Chebyshev polynomials for the integrand: the integration on each subdomain is explicit (see Clenshaw & Curtis 1960).

The computation of $\mathscr{L}_0(\rho)$ is somewhat different since the above decomposition is no longer valid ($\mathscr{K}_0(\rho)$ is not a convergent integral). The difficulty in its numerical calculation lies not only in the oscillations of both exponential and Bessel functions, but also in the fact that the integration domain extends up to infinity. The method we have used combines the Clenshaw & Curtis integration method and an acceleration of convergence (see Evans 1993).

Once $\mathscr{F}(q)$ has been computed, the determination of $p_2^f$ simply consists in a numerical inverse Fourier transform (see (4.10)). Note that both functions $\mathscr{F}(q)$ and $\mathscr{F}(\partial g/\partial z)$ have a circle of singular points with infinite derivatives (due to the slow decay of $q$ and $\partial g/\partial z$ at infinity). We have chosen a numerical integration method

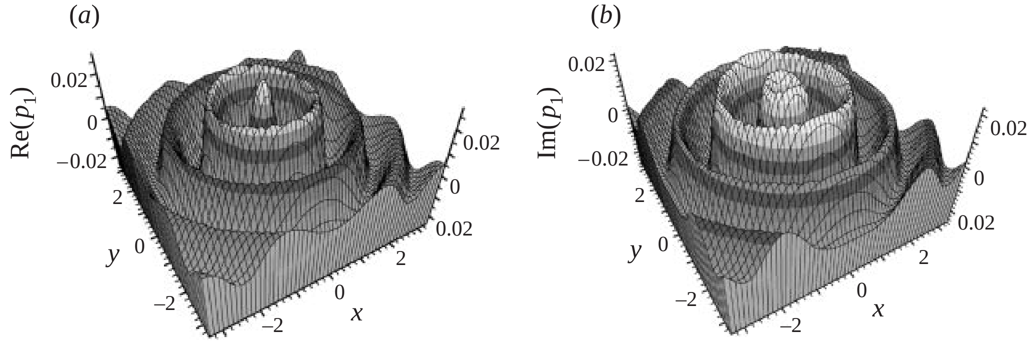


FIGURE 5. (*a*) Real part and (*b*) imaginary part of $p_1$.

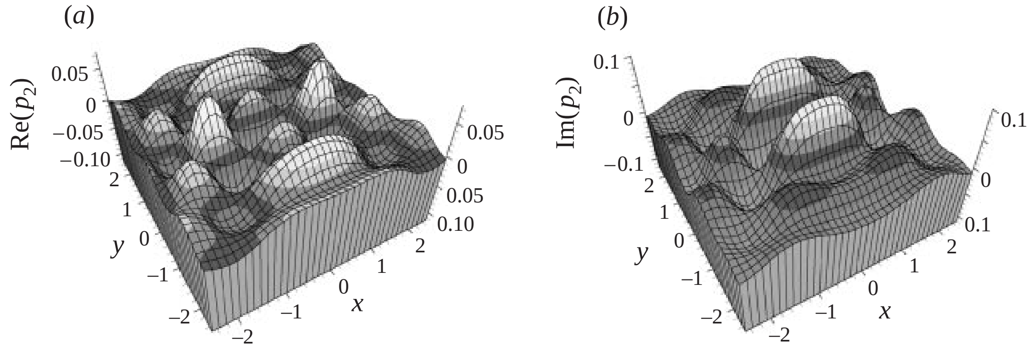


FIGURE 6. (*a*) Real part and (*b*) imaginary part of $p_2$.

containing an adaptative scheme. A more efficient method would be to extract these singularities in order to use a fast Fourier transform algorithm for the computation of the regular part.

### 4.4. *Numerical results*

We suppose that both acoustic and hydrodynamic wavelengths are equal, denoted by $\Lambda$. The body is an ellipsoid of revolution of length $\Lambda/3$ and diameter $\Lambda/30$. Its axis is chosen parallel to the $x$-axis. Its boundary is assumed to vibrate in a time-harmonic way such that the normal velocity is constant on the whole surface at every time (the body 'breathes').

In a first numerical application, the body is immersed at the depth $\Lambda$ (more precisely, its centre is located at point $(0, 0, -\Lambda)$). Figures 5 and 6 represent the real and imaginary parts of the variations of the acoustic pressures $p_1$ and $p_2$, on a horizontal square of approximate side $6\Lambda$ and located at a depth of $\Lambda/2$. Figures 7(*a*) and 7(*b*) corresponds to sections of figures 6(*a*) and 6(*b*) for fixed values of $y$: 0 (solid lines), $\Lambda/2$ (broken lines), $\Lambda$ (mixed lines: dots/dashes), $2\Lambda$ (dotted lines). The first-order pressure seems to be symmetric with respect to the point $(x, y) = (0, 0)$, as for a point source. It actually is not exactly symmetric: the relatively small size of the body (compared with the wavelength $\Lambda$) explains this apparent symmetry. But the real shape of its boundary has a significant influence on the distribution of the pressure field. For the second-order pressure, the particular direction of the swell (which propagates along the $y$-axis) breaks this apparent symmetry: $p_2$ is only symmetric with respect to $x = 0$ and $y = 0$ since this property holds for the body,

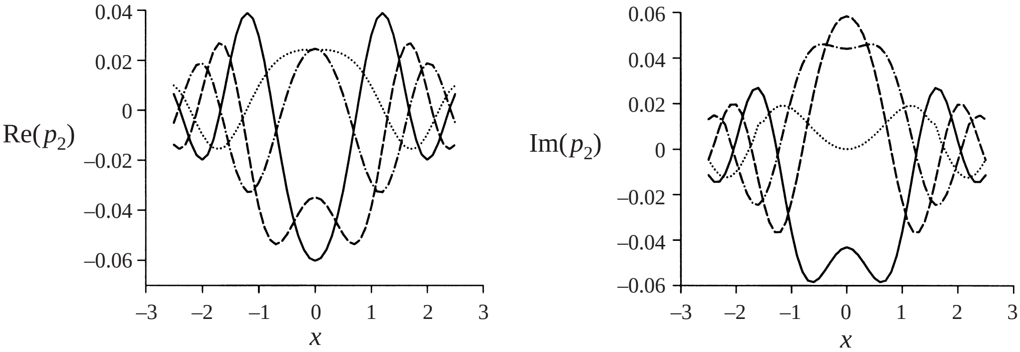


FIGURE 7. ($a$) Real part and ($b$) imaginary part of $p_2$ ($y$ = const).

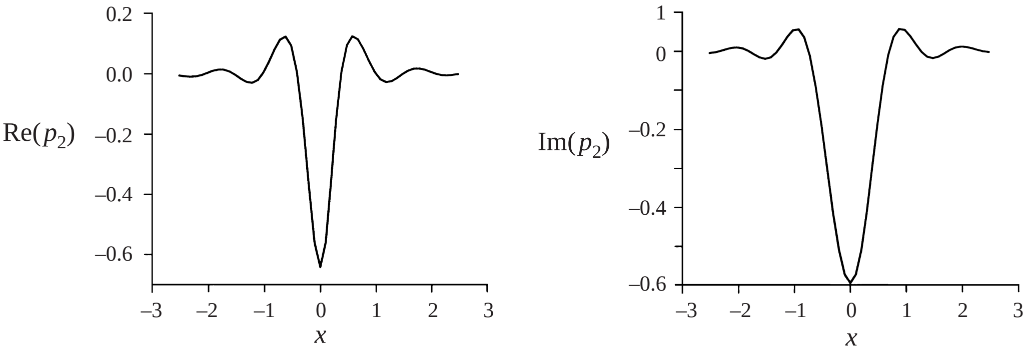


FIGURE 8. ($a$) Real part and ($b$) imaginary part of $p_2$ ($y = 0$).

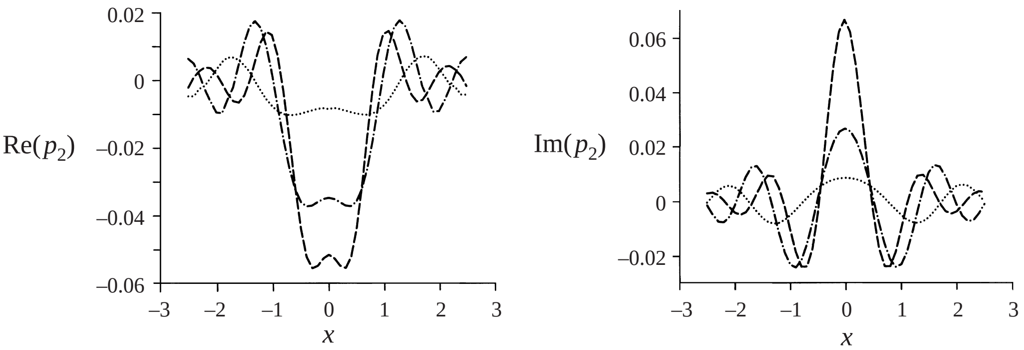


FIGURE 9. ($a$) Real part and ($b$) imaginary part of $p_2$ ($y$ = const).

the swell, as well as $p_1$. Notice that, as one could reasonably expect, the intensities of both pressure fields $p_1$ and $p_2$ have their maxima above the body.

In a second numerical application, the body is immersed at the depth $\Lambda/5$. Figures 8 and 9 show the real and imaginary parts of $p_2$ at $z = -\Lambda/10$ for fixed $y$ (same values and same representations as for figure 7). The fact that the body is now close to the free surface explains the sharp peak located near $x = y = 0$ which appears in figures 8($a$) and 8($b$) (note that the scale is different from figure 9).

In both applications, the 'corrected term' $p_2^c$ is small compared with the 'free pressure' $p_2^f$. This is mainly due to the small diameter of the body (compared with $\Lambda$). Indeed $p_2^f$ takes into account the primary reflection of the first-order pressure field from the corrugated free surface, whereas $p_2^c$ corresponds to a secondary reflection from the body. Here, our body behaves like a 'needle', and the effect of this secondary reflection is negligible (unfortunately, $p_2^f$ is the time-consuming part!). Its contribution may become significant if the diameter of the body and the wavelength are of the same order (and if the body is close enough to the free surface).

## 5. Conclusions

### 5.1. *What has been done*

We have considered a body immersed in an ocean of infinite depth and radiating time-harmonic acoustic waves in the presence of time-harmonic gravity waves. We have restricted our attention to the case when both acoustic and gravity wavelengths are of the same order, which corresponds for usual gravity waves to very low acoustic frequencies. The ocean has been assumed to be a homogeneous and ideal fluid, infinitely deep.

In § 2 it has been shown that the pressure field can be developed as

$$P = P_1 + \mathscr{A} P_2 + \cdots,$$

if $\mathscr{A}$ is the amplitude of the gravity waves ($\mathscr{A} \ll 1$). The first term $P_1$ is the basic acoustic field in the absence of gravity waves, $P_2$ corresponds to its reflection from the perturbed free surface. This term satisfies the Helmholtz equation in the linearized liquid domain, together with a Neumann condition on the hull of the body and a non-homogeneous Dirichlet condition on the mean free surface.

In § 3 we have studied the case of an acoustic point source. In the time-harmonic case all possible solutions $P_2$ have been obtained among which one has been selected, namely the one given by a known distribution of standard normal doublets. Then considering that the source starts at $t = 0$ and working both in space and time for $t \geqslant 0$ it has been completely proved that the solution previously selected was actually the physical one.

In § 4 we have come back to the general case of a vibrating body and presented a splitting method to solve it. The field $p_2$ has been split into a 'free' pressure $p_2^f$ and a correction term $p_2^c$: we were lead by what was done in the point case to give $p_2^f$ in the explicit form of a convolution product in $x$ and $y$, and $p_2^c$ satisfies a standard radiation problem; $p_2^f$ is computed via numerical methods of integration and $p_2^c$ via a coupled BEM/FEM method. A sample of numerical results has been presented.

But it must be pointed out that our splitting method is not neutral but implicitly assumes a choice for the condition at infinity which is only motivated by § 3 but not fully justified.

### 5.2. *A full justification of the method*

As for a point source (§ 3), one can obtain a *limiting-amplitude* result in order to justify the splitting method. But due to the presence of the body, the situation is far more involved and the complete proof requires sharp mathematical tools: it is detailled in Champy-Doutreleau (1998). Here we restrict ourselves to sketching the leading ideas of the proof.

The body is supposed to be at rest for $t < 0$ and to vibrate in a time-harmonic way for $t \geqslant 0$. The corresponding time-dependent pressure field $P_2$ must satisfy the wave equation in the linearized liquid domain together with a time-dependent Dirichlet condition $P_2 = Q$ on the mean free surface, and it must vanish at infinity (in fact it will be non-zero only on a bounded domain for any finite and positive $t$). For this time-dependent problem the previously described splitting (here into $P_2^f$ and $P_2^c$) can be trivially justified. Then the three steps to be performed are the following.

($a$) $Q$ becomes asymptotically time-harmonic. This is a direct consequence of standard limiting-amplitude theorems (see for instance Sanchez-Hubert & Sanchez-Palencia 1989). Such results are closely related to the so-called *limiting-absorption* principle which states that the time-harmonic pressure field appears as the limit of the solution to a dissipative problem when the dissipation vanishes: this dissipation, sometimes referred to as the Rayleigh viscosity, corresponds to complex values of the frequency.

($b$) $P_2^f$ becomes asymptotically time-harmonic. The proof is based on the same arguments as in § 3.2.3: the main difference is that the datum $Q$ is no longer explicitly known, but derives from a time-dependent integral representation formula.

($c$) Finally $P_2^c$ becomes asymptotically a time-harmonic pressure field satisfying the standard radiation condition. This is proved via a refined limiting-amplitude theorem.

## REFERENCES


Bellier, J. L. 1997 Effets du second ordre sur la tenue à la mer d'un sous-marin. PhD thesis, University Paris 6.

Champy-Doutreleau, I. 1998 Analyse et calcul du champ acoustique réfléchi par une surface périodique. PhD thesis, University Paris 6.

Clenshaw, A. & Curtis, A. R. 1960 A method for numerical integration on an automatic computer. *Numerische Mathematik* **2**, 197–205.

Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1954 *Tables of Integral Transforms*, Vol. 2. McGraw-Hill.

Euvrard, D. 1994*a* *Résolution Numérique des Equations aux Dérivées Partielles*. Masson. Paris.

Euvrard, D. 1994*b* About the radiation diagram of an underwater acoustic source in the presence of gravity waves. In *Asymptotic Modelling in Fluid Mechanics* (ed. P. A. Bois *et al.*). Lecture Notes in Physics, vol. 442, pp. 129–144. Springer.

Euvrard, D. & Mechiche Alami, O. 1992 Propagation acoustique sous-marine tridimensionnelle en présence de la houle. *ENSTA Tech. Rep.* 264. Paris.

Evans, G. 1993 *Practical Numerical Integration*. Wiley and Sons.

Friis, A., Grue, J. & Palm, E. 1991 Application of Fourier transform to the second order 2D wave diffraction problem. In *Mathematical Approaches in Hydrodynamics* (ed. T. Miloh), pp. 209–227. SIAM.

Gradshteyn, I. S. & Ryzhik, I. M. 1980 *Table of Integrals, Series and Products* (*corrected and enlarged edition*). Academic.

Jami, A. & Lenoir, M. 1978 A variational formulation for exterior problems in linear hydrodynamics. *Comput. Meth. Appl. Mech. Engng* **16**, 331–359.

Mechiche Alami, O. 1992 Influence de la houle sur le rayonnement acoustique à très basse fréquence d'un corps sous-marin. PhD thesis, University Paris 6. *ENSTA Tech. Rep.* 259. Paris.

Papin, E. 1990 Etude préliminaire théorique en vue du calcul numérique de la tenue à la mer au second ordre. *ENSTA Tech. Rep.* 216. Paris.

Sanchez-Hubert, J. & Sanchez-Palencia E. 1989 *Vibration and Coupling of Continuous Systems, Asymptotic Methods*. Springer.

Schwartz, L. 1965 *Méthodes Mathématiques pour les Sciences Physiques*. Hermann.

Sclavounos, P. D. 1988 Radiation and diffraction of second-order surface waves by floating bodies. *J. Fluid. Mech.* **196**, 65–91.

Treves, F. 1975 *Basic Linear Partial Differential Equations*. Academic.